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Angular Distribution of High Energy Electrons Following Radiation

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U.S. DEPARTMENT OF COMMERCE, Malcolm Baldrige, *Secretary*
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ENERGY-ANGLE DISTRIBUTION OF HIGH ENERGY ELECTRONS FOLLOWING BREMSSTRAHLUNG

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ABSTRACT

We derive an expression for the angular distribution of high energy electrons which have undergone scattering and radiated a photon, integrated over the directions of the emitted photon, in the region of small scattering angles, for which the atomic form factor must be taken into account but the nuclear structure may be neglected. This distribution is analogous to Schiff's high-energy small-angle distribution for photons, integrated over the final electron angles. We show that the correction to the energy-angle distribution of electrons due to atomic screeing is identical in form to the correction to the energy-angle distribution of photons. This correction involves an integral over the atomic form factor, and is evaluated in closed form for the Thomas-Fermi-Molière model. A very simple expression is obtained for the case of complete screening.

Key words: angular distribution of scattered electrons; atomic screening effects in electron scattering; bremsstrahlung; high energy electron scattering; small angle electron scattering cross section; Schiff energy-angle distribution.

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CONTENTS

INTRODUCTION

In this note we derive an expression for the angular distribution of high energy electrons which have undergone scattering and radiated a photon, integrated over the directions of the emitted photon. This cross section has been obtained previously under conditions pertinent to large angle, large momentum transfer electron scattering experiments [1-3]. Here we are concerned with this same cross section, but under conditions appropriate to a photon tagging system, for which the important contribution comes from small angle scatterings, corresponding to momentum transfers $q \lesssim O(mc)$. Although momentum transfers $q > O(mc)$ give a negligible contribution if one subsequently also integrates over the angles of the scattered electron, our angular distribution for the scattered electron is in fact valid for the wider range of momentum transfers, $q_m \leqq q << \frac{1}{R}$, where q_m is the minimum possible momentum transfer and R is the nuclear radius, i.e., it is valid provided that the nuclear form factor does not enter significantly. We therefore neglect target recoil, and in place of the nuclear form factor which appears as $\mathcal{F}^2(q)$ in the completely differential cross section involving large momentum transfers, we now have the atomic form factor, appearing as $[1-F(q)]^2$. (See [1], p. B1345, comments following (5).) We start therefore with the Bethe-Heitler (Born approximation) differential cross section for bremsstrahlung. We note that the integration of this cross section over the angles of the final electron, assuming small momentum transfers, is well-known [4], and in the case of complete screening results in the Schiff energy-angle distribution [4,5].

The integration of the bremsstrahlung cross section over final electron directions has also been performed [6], including Coulomb corrections, in the region of small momentum transfers $q \lesssim O(mc)$, and some of the analysis given in [6] will be of use to us in the present work.

However, we have not found, in the available literature, an expression for the bremsstrahlung cross section integrated over photon directions that is useful for the region of small momentum transfers (the equivalent of Schiff's expression, but integrated over photon directions rather than final electron directions). This lack is quite understandable. The major interest in the cross section for small angles (low momentum transfers) has been to obtain the angular distribution of bremsstrahlung (integrated over electron directions) since the photons are emitted preferentially in the forward direction. On the other hand, when the integration over photon directions has been performed, it has been in connection with high energy electron scattering experiments, which are performed at large angles and large momentum transfers in order to obtain nuclear structure information.

II. THE CROSS SECTION INTEGRATED OVER PHOTON DIRECTIONS

We turn now to the Bethe-Heitler (Born approximation) cross section integrated over photon angles, but with no approximations concerning the energies of the incident and final electron or the angles of the scattered electron. This has been given by Maximon and Isabelle [1] in connection with the radiative tail in elastic electron scattering (eqs (4) and (5) on p. B1345 of [1]). We need only replace the nuclear structure function $\mathcal{F}(q)$ in eq (4) of [1] by $1-F(q)$, where $F(q)$ is the atomic form factor. We then have*

$$\frac{d\sigma}{d\Omega} = \frac{1}{2\pi}\frac{e^2}{\hbar c}\left(\frac{Ze^2}{mc^2}\right)^2 \frac{p_2}{p_1}\frac{d\epsilon_2}{k}\int_{q_m^2}^{q_M^2}\frac{[1-F(q)]^2}{q^4}\left\{\quad\right\} d(q^2) \tag{1}$$

where

---

*The most direct way of arriving at the result given in (2) is the procedure used in [3] (rather than that followed in [1]). In [3] the integration over photon angles is performed in a coordinate system with z-axis in the direction of $\underline{p}_1-\underline{p}_2$, in which the photon angles are θ_k, ϕ_k. The momentum transfer to the atom,

$$q^2 = (\underline{p}_1-\underline{p}_2-\underline{k})^2 = (\underline{p}_1-\underline{p}_2)^2 + k^2 - 2k|\underline{p}_1-\underline{p}_2|\cos\theta_k$$

is independent of the azimuthal angle, ϕ_k, of the photon. The integration over ϕ_k is then straightforward and leads to the expression given below in (2), while the integration over $\cos\theta_k$ may be written as an integration over q^2, as in (1).

Note that we have replaced dk in [1] by $d\epsilon_2$. In the absence of recoil this makes no difference since then $k = \epsilon_1-\epsilon_2$ and hence $dk = -d\epsilon_2$. However, since in the present investigation the final electron is observed, we find it more reasonable to use $d\epsilon_2$.

$$\left\{\quad\right\} = -\frac{2k}{(2\lambda + k^2)^{\frac{1}{2}}}$$

$$- k\left(\frac{1}{D_1^{\frac{1}{2}}} - \frac{1}{D_2^{\frac{1}{2}}}\right)\left(\frac{q^4 + 4\lambda^2 - 4q^2(\epsilon_1{}^2+\epsilon_2{}^2-1) - 16\epsilon_1\epsilon_2}{2\lambda - q^2}\right)$$

$$+ 2k\,\frac{(4\epsilon_1^2 - q^2)}{D_1{}^{3/2}}\left[2\lambda(\lambda-k\epsilon_2) - (\lambda+k\epsilon_1)q^2\right]$$

$$- 2k\,\frac{(4\epsilon_1^2 - q^2)}{D_2{}^{3/2}}\left[2\lambda(\lambda+k\epsilon_1) - (\lambda-k\epsilon_2)q^2\right] \qquad (2)$$

with

$$\lambda = \epsilon_1\epsilon_2 - p_1p_2\cos\vartheta - 1 = \tfrac{1}{2}(|\underline{p}_1 - \underline{p}_2|^2 - k^2)$$

$$\lambda_0 = \epsilon_1\epsilon_2 - p_1p_2 - 1$$

$$D_1 = \{p_1(q^2 - q_2{}^2) + 2p_2\lambda_0\cos\vartheta\}^2 + 4k^2p_2{}^2\sin^2\vartheta$$

$$D_2 = \{p_2(q^2 - q_1{}^2) + 2p_1\lambda_0\cos\vartheta\}^2 + 4k^2p_1{}^2\sin^2\vartheta$$

$$q_m = |\underline{p}_1 - \underline{p}_2| - k = (2\lambda + k^2)^{\frac{1}{2}} - k$$

$$q_M = |\underline{p}_1 - \underline{p}_2| + k = (2\lambda + k^2)^{\frac{1}{2}} + k$$

$$q_2 = 2p_2\sin\tfrac{1}{2}\vartheta$$

$$q_1 = 2p_1\sin\tfrac{1}{2}\vartheta$$

$$\delta = p_1 - p_2 - k$$

$$k = \epsilon_1 - \epsilon_2 \quad . \qquad (3)$$

Here $d\Omega = \sin\vartheta \, d\vartheta \, d\phi$ refers to the final electron, with polar and azimuthal angles ϑ, ϕ in a coordinate system with z-axis in the direction of the initial electron.

ϵ_1, $\underline{p}_1$ are the energy and momentum of the incident electron.

ϵ_2, $\underline{p}_2$ are the energy and momentum of the final electron.

$k = \epsilon_1 - \epsilon_2$ is the energy of the emitted photon.

All of the energies and momenta are measured in the laboratory system, in units of mc^2 and mc, respectively.

Having performed the integration over photon directions, giving (1) and (2), we are now confronted by a rather complicated looking expression which cannot be integrated analytically because of the presence of the screening function, which is in general only given numerically. However, if we consider the screening correction rather than the cross section itself, then a considerable simplification results: For the cross section with screening we write

$$d\sigma_{scr} = \left(d\sigma_{scr} - d\sigma_{unscr}\right) + d\sigma_{unscr}$$

$$\equiv \Delta\sigma_{scr} + d\sigma_{unscr} \quad . \tag{4}$$

In the integral in (1) we now write

$$(1-F)^2 = \left[(1-F)^2 - 1\right] + 1 \quad . \tag{5}$$

The term $[(1-F)^2 - 1]$ gives the screening correction, $\Delta\sigma_{scr}$; the last term, 1, gives the unscreened cross section, $d\sigma_{unscr}$. This separation, which was used in [6] in integrating over the angles of the final

electron, has two important consequences. First, the integration required to obtain the unscreened cross section can then be performed in closed form. In the present case, in which we integrate over photon angles, the unscreened cross section is given in [1] (p. B1346, eq (6)). Second, in the screening correction only very small values of q are significant, viz.,

$$q \lesssim O(\beta) \quad , \tag{6}$$

where β is the inverse atomic screening radius. (In the Thomas-Fermi model, $\beta \approx \frac{Z^{1/3}}{121} \ll 1$.) This may be seen by noting that we can write, approximately ([6], p. 897, eq (6.30))

$$\frac{1 - F}{q^2} = \frac{1}{q^2 + \beta^2} \tag{7}$$

and hence

$$\frac{(1 - F)^2 - 1}{q^4} = - \frac{2q^2\beta^2 + \beta^4}{(q^2 + \beta^2)^2 q^4} \quad . \tag{8}$$

Thus in the integral for the screening correction, the contribution from $q \gtrsim O(1)$ is of order β^2 and may be neglected. The only non-negligible contribution to the integral comes from $q \lesssim O(\beta)$ and, as we will show, a remarkable simplification of the integrand in (1) results in this region of very small momentum transfers. In fact, we find that the screening correction to the angular distribution of the scattered electron (integrated over photon angles) is the same as the previously obtained [6]

screening correction to the angular distribution of emitted photons (integrated over final electron angles), once one has made the appropriate change of variable from photon scattering angle to electron scattering angle.

The explicit expressions for the screening correction and the unscreened cross section (the angular distribution of scattered electrons, integrated over photon angles) derived in this report are given in the text by eqs (61), (66), and (67) (for the correction for intermediate screening), by eqs (61), (66), and (71) (for the correction for complete screening, and by eqs (63a) and (63b) for the unscreened cross section.

III. TECHNICAL PRELIMINARIES

We first make a few preliminary observations concerning the variables defined in (3) which are useful for the simplification of the integrand, (2), for the screening correction. In order to simplify the order of magnitude observations that follow, we will not only assume high energies of both the incident and final electron,

$$\epsilon_1 \gg 1 \quad , \quad \epsilon_2 \gg 1 \tag{9}$$

but will assume as well that the photon energy is of the same order of magnitude as these energies:

$$k = O(\epsilon) \quad . \tag{10}$$

A more detailed analysis is required to show that our final results are valid for somewhat smaller photon energies, but we will not present that here.

In the following analysis we show that, as a consequence of (9) and (10), the variables in (3) must satisfy certain restrictive conditions in order that we have a non-negligible screening correction. A very significant simplification of the integrand then follows as a result of these conditions, which are

$$\text{i)} \quad \epsilon \gtrsim O(1/\beta)$$

$$\text{ii)} \quad \vartheta^2 \lesssim O(\beta/\epsilon) \lesssim O(\beta^2)$$

$$\text{iii)} \quad O(1) \lesssim \lambda \lesssim O(\epsilon\beta) \quad .$$

From iii) it follows, then, since the significant values of q in the integrand for the screening correction are $q \lesssim O(\beta)$, that we can set

$$\frac{q^2}{\lambda} \lesssim O(\beta^2) << 1 \tag{11}$$

in the integrand. Most importantly, this permits us to expand the denominators $D_1^{1/2}$, $D_2^{1/2}$, $D_1^{3/2}$, and $D_2^{3/2}$ in powers of $\frac{q^2}{\lambda}$, as a result of which we find a very simple form for the integrand.

From (3), the minimum momentum transfer, q_m, is given by

$$q_m = |\underline{p}_1 - \underline{p}_2| - k \geqq p_1 - p_2 - k \equiv \delta$$

and from (9) and (10) (see appendix)

$$\delta \approx \frac{k}{2\epsilon_1 \epsilon_2} = O(1/\epsilon) \quad .$$

Now if we are to have a non-negligible screening correction, we must have

$$q_m \lesssim O(\beta) \tag{12}$$

and hence also $\delta \lesssim O(\beta)$, from which we have restriction (i),

$$\epsilon \gtrsim O(1/\beta) \quad . \tag{13}$$

Next, from (3) we also have

$$q_m = (2\lambda + k^2)^{\frac{1}{2}} - k$$

$$= \frac{2\lambda}{(2\lambda + k^2)^{\frac{1}{2}} + k} \quad . \tag{14}$$

Thus from (10) we have $(2\lambda + k^2)^{\frac{1}{2}} + k = O(\epsilon)$ and $q_m = O(\lambda/\epsilon)$. Again in order that $q_m \lesssim O(\beta)$ we now must require

$$\lambda \lesssim O(\epsilon\beta) \quad . \tag{15}$$

But at high energies we have (see appendix)

$$\lambda \approx \frac{k^2}{2\epsilon_1\epsilon_2} + 2\epsilon_1\epsilon_2 \sin^2 \tfrac{1}{2}\vartheta$$

$$= O(1) + O(\epsilon^2\vartheta^2) \gtrsim O(1) \quad . \tag{16}$$

Now from (15) and (16) we have

$$\epsilon^2\vartheta^2 \lesssim O(\epsilon\beta) \quad ,$$

which, together with (11), gives (ii):

$$\vartheta^2 \lesssim O(\beta/\epsilon) \lesssim O(\beta^2) \quad . \tag{17}$$

Again from (15) and (16),

$$O(1) \lesssim \lambda \lesssim O(\epsilon\beta) \tag{18}$$

which is (iii).

As a prelude to writing the explicit expression for the expansion of the integrand ($\{\ \}$ in (2)) in powers of q^2/λ, we examine the order of magnitude of each of the terms there. We show that, individually, the largest terms there are each of order $\frac{k^2}{\lambda^2} \gtrsim O(1/\beta^2) >> 1$. However, taken together they cancel each other almost completely, and as a consequence we must keep the terms of order 1 in $\{\ \}$, these being of order β^2 relative to the individually large terms. To see this clearly we first write the expressions for D_1 and D_2 in a form more suitable for an expansion in powers of q^2/λ. (The expressions given in (3) are taken directly from [1].) From the appendix of the present report we have

$$D_1 = 4p_2{}^2\lambda^2 - 4[(\epsilon_1\epsilon_2 - 1)\lambda - k^2]q^2 + p_1{}^2q^4 \tag{19a}$$

and

$$D_2 = 4p_1{}^2\lambda^2 - 4[(\epsilon_1\epsilon_2 - 1)\lambda - k^2]q^2 + p_2{}^2q^4 \quad . \tag{19b}$$

Now from (11) we note that in (19a) and (19b) the second term is of order

$$\frac{q^2}{\lambda} \lesssim O(\beta^2)$$

relative to the first term, and the third term is of order

$$(q^2/\lambda)^2 \lesssim O(\beta^4)$$

relative to the first term. Thus we can write

$$D_1^{\frac{1}{2}} = 2p_2\lambda(1 + O(\beta^2)) \tag{20a}$$

$$D_2^{\frac{1}{2}} = 2p_1\lambda(1 + O(\beta^2)) \quad . \tag{20b}$$

Again referring to (2), we note from (11) and (18) that

$$\left(\frac{q^4 + 4\lambda^2 - 4q^2(\epsilon_1{}^2+\epsilon_2{}^2-1) - 16\epsilon_1\epsilon_2}{2\lambda - q^2}\right) = \frac{-16\epsilon_1\epsilon_2}{2\lambda}\left(1 + O(\beta^2)\right) \tag{21}$$

$$\left[2\lambda(\lambda - k\epsilon_2) - (\lambda + k\epsilon_1)q^2\right] = 2\lambda(\lambda - k\epsilon_2)\left(1 + O(\beta^2)\right) \tag{22}$$

$$\left[2\lambda(\lambda + k\epsilon_1) - (\lambda - k\epsilon_2)q^2\right] = 2\lambda(\lambda + k\epsilon_1)\left(1 + O(\beta^2)\right) . \tag{23}$$

Furthermore, in (22) and (23) we can write, from (3),

$$\begin{aligned} \lambda - k\epsilon_2 &= \epsilon_1\epsilon_2 - p_1p_2\cos\vartheta - 1 - \epsilon_1\epsilon_2 + \epsilon_2{}^2 \\ &= p_2{}^2 - p_1p_2\cos\vartheta \\ &= -p_2(p_1-p_2) + 2p_1p_2\sin^2\tfrac{1}{2}\vartheta \end{aligned} \tag{24}$$

and

$$\begin{aligned} \lambda + k\epsilon_1 &= \epsilon_1\epsilon_2 - p_1p_2\cos\vartheta - 1 + \epsilon_1{}^2 - \epsilon_1\epsilon_2 \\ &= p_1{}^2 - p_1p_2\cos\vartheta \\ &= p_1(p_1-p_2) + 2p_1p_2\sin^2\tfrac{1}{2}\vartheta \quad . \end{aligned} \tag{25}$$

Now for high energies, $p_1-p_2 \approx \epsilon_1-\epsilon_2 = k$. Thus, assuming $k = O(\epsilon)$ and noting that $\vartheta^2 \lesssim O(\beta/\epsilon)$ from (17), we have

$$\lambda - k\epsilon_2 = -p_2(p_1-p_2)\left(1 + O(\beta/\epsilon)\right) \tag{26}$$

$$\lambda + k\epsilon_1 = p_1(p_1-p_2)\left(1 + O(\beta/\epsilon)\right) \quad . \tag{27}$$

We now collect the terms in $\Big\{ \; \Big\}$ with a power of D_1 or D_2 in the denominator. From (20a), (20b), and (21) we have, for the terms with $D_1^{-\frac{1}{2}}$ and $D_2^{-\frac{1}{2}}$:

$$-k\left(\frac{1}{D_1^{\frac{1}{2}}} - \frac{1}{D_2^{\frac{1}{2}}}\right)\left(\frac{q^4+4\lambda^2-4q^2(\epsilon_1{}^2+\epsilon_2{}^2-1)-16\epsilon_1\epsilon_2}{2\lambda - q^2}\right) = \frac{4k\epsilon_1\epsilon_2}{2\lambda^2}\left(\frac{1}{p_2} - \frac{1}{p_1}\right)\left(1+O(\beta^2)\right) \tag{28}$$

$$= \frac{4k(p_1-p_2)}{\lambda^2}\,\frac{\epsilon_1\epsilon_2}{p_1p_2}\,\left(1 + O(\beta^2)\right) \; .$$

Next, substituting (26) and (27) in (22) and (23) and using (20a) and (20b), we have, for the terms with $D_1^{-3/2}$ and $D_2^{-3/2}$:

$$\frac{2k(4\epsilon_2{}^2-q^2)}{D_1{}^{3/2}}\left[2\lambda(\lambda-k\epsilon_2) - (\lambda+k\epsilon_1)q^2\right] = \frac{-2k(p_1-p_2)}{\lambda^2}\,\frac{\epsilon_2{}^2}{p_2{}^2}\,\left(1 + O(\beta^2)\right) \tag{29}$$

and

$$\frac{-2k(4\epsilon_1{}^2-q^2)}{D_2{}^{3/2}}\left[2\lambda(\lambda+k\epsilon_1) - (\lambda-k\epsilon_2)q^2\right] = \frac{-2k(p_1-p_2)}{\lambda^2}\,\frac{\epsilon_1{}^2}{p_1{}^2}\,\left(1 + O(\beta^2)\right) \; . \tag{30}$$

Thus the large terms in $\left\{ \ \right\}$, those exhibited specifically in (28), (29), and (30), are indeed each of order $\frac{k^2}{\lambda^2} \gtrsim 0 1/\beta^2) \gg 1$, as mentioned. However, when added they give

$$\frac{2k(p_1-p_2)}{\lambda^2}\left(\frac{\epsilon_1}{p_1}-\frac{\epsilon_2}{p_2}\right)^2 = 0\left(\frac{k^2}{\lambda^2}\cdot\frac{k^2}{\epsilon^6}\right) \tag{31}$$

since, as shown in the appendix,

$$\left(\frac{\epsilon_1}{p_1}-\frac{\epsilon_2}{p_2}\right)^2 = 0\left(\frac{k^2}{\epsilon^6}\right) . \tag{32}$$

Thus the "large" terms cancel, leaving a remainder of order

$$\frac{k^4}{\lambda^2\epsilon^6} = \left(\frac{k^2}{\lambda\epsilon^2}\right)^2\frac{1}{\epsilon^2} \lesssim \frac{1}{\epsilon^2}$$

since from (16) $\frac{\lambda\epsilon^2}{k^2} \gtrsim 0(1)$. This remainder can therefore be neglected; the contribution to be retained thus comes from the terms of $0(1)$, i.e., from the terms indicated in (20a) - (23) as being of $0(\beta^2)$ relative to the individually large terms. It should be noted that the very significant cancellation, giving the factor

$$\left(\frac{\epsilon_1}{p_1}-\frac{\epsilon_2}{p_2}\right)^2 = 0\left(\frac{k^2}{\epsilon^6}\right) \lesssim 0(1/\epsilon^4)$$

results from our particular choice for the "large" terms in (28)-(30). Had we made a less auspicious choice, for example by writing

$$\lambda - k\epsilon_2 = -k\epsilon_2(1 + O(\beta/\epsilon))$$

$$\lambda + k\epsilon_1 = k\epsilon_1(1 + O(\beta/\epsilon))$$

instead of (26) and (27), then the large terms would only have cancelled to relative order $1/\epsilon^2$ (rather than to relative order $1/\epsilon^4$). The remainder would then have been of order 1, and have had to be retained.

IV. DERIVATION OF THE SCREENING CORRECTION

We now proceed with the expansion of the terms in $\Big\{ \ \Big\}$ in (2), retaining all terms of O(1), i.e., of order β^2 relative to the large terms which have just been shown to cancel. However, we neglect terms which are individually of $O(\beta^2)$ or of $O(1/\epsilon^2)$; these terms are of order β^4 or of order β^2/ϵ^2 relative to the large terms.

Referring to (2), we have first

$$\frac{-2k}{(2\lambda + k^2)^{\frac{1}{2}}} = \frac{-2}{\left(1 + \frac{2\lambda}{k^2}\right)^{\frac{1}{2}}}$$

$$= -2 + O(\lambda/k^2) \quad . \tag{33}$$

Here, from (18), (10), and (13),

$$\frac{\lambda}{k^2} \lesssim O(\beta/\epsilon) \lesssim O(\beta^2) \quad . \tag{34}$$

Thus, as just explained, we retain the term -2, but neglect the term of $O(\lambda/k^2) \lesssim O(\beta^2)$, writing

$$\frac{-2k}{(2\lambda + k^2)^{\frac{1}{2}}} = -2\left(1 + O(\beta^2)\right) \quad . \tag{35}$$

Next, in the term with factors $D_1^{-\frac{1}{2}}$ and $D_2^{-\frac{1}{2}}$ we expand the factors

$$\left(\frac{q^4+4\lambda^2-4q^2(\epsilon_1{}^2+\epsilon_2{}^2-1)-16\epsilon_1\epsilon_2}{2\lambda - q^2}\right)$$

$$= \left[-\frac{8\epsilon_1\epsilon_2}{\lambda} - \frac{2q^2(\epsilon_1{}^2+\epsilon_2{}^2)}{\lambda} + 2\lambda - \frac{4\epsilon_1\epsilon_2 q^2}{\lambda^2}\right](1+O(\beta^4)) \quad (36)$$

and, from (19a) and (19b),

$$-k\left(\frac{1}{D_1{}^{\frac{1}{2}}} - \frac{1}{D_2{}^{\frac{1}{2}}}\right) = -\frac{k}{2p_2\lambda}\left[1 + \frac{1}{2}\frac{(\epsilon_1\epsilon_2\lambda - k^2)q^2}{p_2{}^2\lambda^2}\right](1 + O(\beta^4))$$

$$+\frac{k}{2p_1\lambda}\left[1 + \frac{1}{2}\frac{(\epsilon_1\epsilon_2\lambda - k^2)q^2}{p_1{}^2\lambda^2}\right](1 + O(\beta^4))$$

$$= -\frac{k}{2\lambda}\frac{(p_1-p_2)}{p_1p_2}\left[1 + \frac{k(p_1{}^2+p_1p_2+p_2{}^2)q^2}{2p_1{}^2p_2{}^2\lambda^2}\right](1 + O(\beta^4)) \quad . (37)$$

Multiplying the factors in (36) and (37) then gives the expansion of the term in (2) with factors $D_1{}^{-\frac{1}{2}}$ and $D_2{}^{-\frac{1}{2}}$:

$$-k\left(\frac{1}{D_1^{\frac{1}{2}}} - \frac{1}{D_2^{\frac{1}{2}}}\right)\left(\frac{q^4+4\lambda^2-4q^2(\epsilon_1{}^2+\epsilon_2{}^2-1)-16\epsilon_1\epsilon_2}{2\lambda - q^2}\right)$$

$$= - \frac{k(p_1-p_2)}{2\lambda p_1 p_2}\left[1 + \frac{(\epsilon_1\epsilon_2\lambda-k^2)(p_1{}^2+p_1p_2+p_2{}^2)q^2}{2p_1{}^2p_2{}^2\lambda^2}\right]$$

$$\times\left[- \frac{8\epsilon_1\epsilon_2}{\lambda} - \frac{2q^2(\epsilon_1{}^2+\epsilon_2{}^2)}{\lambda} + 2\lambda - \frac{4\epsilon_1\epsilon_2 q^2}{\lambda^2}\right](1+O(\beta^4))$$

$$= \frac{4k(p_1-p_2)}{2\lambda}\frac{\epsilon_1\epsilon_2}{p_1p_2}$$

$$- \frac{k(p_1-p_2)}{p_1p_2}$$

$$+ \frac{k(p_1-p_2)q^2}{p_1p_2\lambda}\left[\frac{2\epsilon_1\epsilon_2(\epsilon_1\epsilon_2\lambda-k^2)(p_1{}^2+p_1p_2+p_2{}^2)}{p_1{}^2p_2{}^2\lambda^3} + \frac{(\epsilon_1{}^2+\epsilon_2{}^2)}{\lambda} + \frac{2\epsilon_1\epsilon_2}{\lambda^2}\right]$$

$$+ \text{ terms of relative order } \beta^4 \quad . \tag{38}$$

Finally, we have the terms in (2) with factors $D_1^{-3/2}$ and $D_2^{-3/2}$. Again we expand, substituting (24) and (25) in the factors in these terms:

$$2\lambda(\lambda-k\epsilon_2) - (\lambda+k\epsilon_1)q^2 = 2\lambda[-p_2(p_1-p_2) + 2p_1p_2\sin^2\tfrac{1}{2}\vartheta]$$

$$- [p_1(p_1-p_2) + 2p_1p_2\sin^2\tfrac{1}{2}\vartheta]q^2$$

$$= [-2\lambda p_2(p_1-p_2) + 4\lambda p_1p_2\sin^2\tfrac{1}{2}\vartheta$$

$$- p_1(p_1-p_2)q^2]\,(1+O(\beta^2/\epsilon^2)) \tag{39a}$$

$$2\lambda(\lambda+k\epsilon_1) - (\lambda-k\epsilon_2)q^2 = 2\lambda[p_1(p_1-p_2) + 2p_1p_2\sin^2\tfrac{1}{2}\vartheta]$$

$$- [-p_2(p_1-p_2) + 2p_1p_2\sin^2\tfrac{1}{2}\vartheta]q^2$$

$$= [2\lambda p_1(p_1-p_2) + 4\lambda p_1p_2\sin^2\tfrac{1}{2}\vartheta$$

$$+ p_2(p_1-p_2)q^2]\,(1+O(\beta^2/\epsilon^2)) \quad . \tag{39b}$$

We then expand $D_1^{-3/2}$ and $D_2^{-3/2}$, using (19a) and (19b), from which

$$D_1^{-3/2} = \frac{1}{(2p_2\lambda)^3}\left[1 + \frac{3}{2}\,\frac{(\epsilon_1\epsilon_2\lambda-k^2)q^2}{p_2{}^2\lambda^2}\right](1+O(\beta^4)) \tag{40}$$

$$D_2^{-3/2} = \frac{1}{(2p_1\lambda)^3}\left[1 + \frac{3}{2}\,\frac{(\epsilon_1\epsilon_2\lambda-k^2)q^2}{p_1{}^2\lambda^2}\right](1+O(\beta^4)) \tag{41}$$

Then, referring to the terms in (2), we have, from (39a) and (40),

$$2k \frac{(4\epsilon_2{}^2-q^2)}{D_1{}^{3/2}} \left[2\lambda(\lambda-k\epsilon_2) - (\lambda+k\epsilon_1)q^2\right]$$

$$= \frac{k\epsilon_2{}^2}{p_2{}^3\lambda^3}\left[1 + \frac{3}{2}\frac{(\epsilon_1\epsilon_2\lambda-k^2)}{p_2{}^2\lambda^2} q^2\right]$$

$$\times \left[-2\lambda p_2(p_1-p_2) + 4\lambda p_1p_2\sin^2\tfrac{1}{2}\vartheta - p_1(p_1-p_2)q^2\right] (1+O(\beta^4))$$

$$= - \frac{2k(p_1-p_2)}{\lambda^2}\frac{\epsilon_2{}^2}{p_2{}^2}$$

$$+ \frac{4kp_1}{\lambda^2}\frac{\epsilon_2{}^2}{p_2{}^2}\sin^2\tfrac{1}{2}\vartheta$$

$$- \frac{k(p_1-p_2)}{\lambda^4}\frac{\epsilon_2{}^2}{p_2{}^4} q^2\left[3(\epsilon_1\epsilon_2\lambda-k^2) + p_1p_2\lambda\right]$$

$$+ \text{ terms of relative order } \beta^4. \tag{42}$$

Finally, from (39b) and (41) we have

$$-2k \frac{(4\epsilon_1{}^2-q^2)}{D_2{}^{3/2}} \left[2\lambda(\lambda+k\epsilon_1) - (\lambda-k\epsilon_2)q^2\right]$$

$$= -\frac{k\epsilon_1{}^2}{p_1{}^3\lambda^3}\left[1 + \frac{3}{2}\frac{(\epsilon_1\epsilon_2\lambda-k^2)}{p_1{}^2\lambda^2}q^2\right]$$

$$\times \left[2\lambda p_1(p_1-p_2) + 4\lambda p_1 p_2 \sin^2\tfrac{1}{2}\vartheta + p_2(p_1-p_2)q^2\right] (1+O(\beta^4))$$

$$= -\frac{2k(p_1-p_2)}{\lambda^2}\frac{\epsilon_1{}^2}{p_1{}^2}$$

$$-\frac{4kp_2}{\lambda^2}\frac{\epsilon_1{}^2}{p_1{}^2}\sin^2\tfrac{1}{2}\vartheta$$

$$-\frac{k(p_1-p_2)}{\lambda^4}\frac{\epsilon_1{}^2}{p_1{}^4}q^2\left[3(\epsilon_1\epsilon_2\lambda-k^2) + p_1p_2\lambda\right]$$

$$+ \text{ terms of relative order } \beta^4. \tag{43}$$

Now the first term on the right hand side of each of the three expressions — (38), (42), and (43) — are the large terms we have already considered (note eqs. (28), (29), and (30)). As mentioned, they are individually of order $k^2/\lambda^2 = O(\epsilon^2)$, but when added give a contribution of $O(1/\epsilon^2)$, which we neglect (see eqs (31) and (32)). We are thus left with the remaining terms, of $O(1)$, in (38), (42), and (43), as well as the term -2 from eqs (33)-(35). We first gather the terms without a factor q^2. These come from (35) and the second term on the right hand side of eqs (38), (42), and (43). They are

$$-2 - \frac{k(p_1-p_2)}{p_1p_2} + \frac{4kp_1}{\lambda^2}\frac{\epsilon_2{}^2}{p_2{}^2}\sin^2\tfrac{1}{2}\vartheta - \frac{4kp_2}{\lambda^2}\frac{\epsilon_1{}^2}{p_1{}^2}\sin^2\tfrac{1}{2}\vartheta \quad . \tag{44}$$

Since these terms are each of O(1), we may now set $\epsilon_1 \approx p_1$, $\epsilon_2 \approx p_2$ throughout, neglecting terms of $O(1/\epsilon^2)$. We then have

$$-2 - \frac{k^2}{\epsilon_1\epsilon_2} + \frac{4k^2}{\lambda^2}\sin^2\tfrac{1}{2}\vartheta = -\frac{(\epsilon_1{}^2+\epsilon_2{}^2)}{\epsilon_2\epsilon_2} + \frac{4k^2}{\lambda^2}\sin^2\tfrac{1}{2}\vartheta \quad . \tag{45}$$

We next gather the terms with a factor q^2. These come from the last term on the right hand side of eqs (38), (42), and (43). They are

$$\frac{k(p_1-p_2)}{\lambda p_1p_2}\,q^2\left[\frac{2\epsilon_1\epsilon_2(\epsilon_1\epsilon_2\lambda-k^2)(p_1{}^2+p_1p_2+p_2{}^2)}{p_1{}^2p_2{}^2\lambda^3} + \frac{(\epsilon_1{}^2+\epsilon_2{}^2)}{\lambda} + \frac{2\epsilon_1\epsilon_2}{\lambda^2}\right]$$

$$-\frac{k(p_1-p_2)}{\lambda^4}\,q^2\,\frac{\epsilon_2{}^2}{p_2{}^4}\left[3(\epsilon_1\epsilon_2\lambda - k^2) + p_1p_2\lambda\right]$$

$$-\frac{k(p_1-p_2)}{\lambda^4}\,q^2\,\frac{\epsilon_1{}^2}{p_1{}^4}\left[3(\epsilon_1\epsilon_2\lambda - k^2) + p_1p_2\lambda\right] \quad . \tag{46}$$

As with the terms without a factor q^2, listed in (44), we may now set $\epsilon_1 \approx p_1$, $\epsilon_2 \approx p_2$ throughout, again neglecting terms of $O(1/\epsilon^2)$. The terms in (46) then become

$$\frac{k^2q^2}{\lambda^2}\left[\begin{aligned}&\frac{2(\epsilon_1\epsilon_2\lambda-k^2)(\epsilon_1{}^2+\epsilon_1\epsilon_2+\epsilon_2{}^2)}{\epsilon_1{}^2\epsilon_2{}^2\lambda^2}+\frac{(\epsilon_1{}^2+\epsilon^2)}{\epsilon_1\epsilon_2}+\frac{2}{\lambda}\\&-\frac{3(\epsilon_1\epsilon_2\lambda-k^2)}{\epsilon_2{}^2\lambda^2}-\frac{3(\epsilon_1\epsilon_2\lambda-k^2)}{\epsilon_1{}^2\lambda^2}-\frac{1}{\lambda}\left(\frac{\epsilon_1}{\epsilon_2}+\frac{\epsilon_2}{\epsilon_1}\right)\end{aligned}\right]$$

$$=\frac{k^2q^2}{\lambda^2}\left[\begin{aligned}&\frac{(\epsilon_1\epsilon_2\lambda-k^2)}{\lambda^2}\left[\frac{2}{\epsilon_2{}^2}+\frac{2}{\epsilon_1{}^2}+\frac{2}{\epsilon_1\epsilon_2}-\frac{3}{\epsilon_1{}^2}-\frac{3}{\epsilon_2{}^2}\right]\\&+\frac{2}{\lambda}+\frac{(\epsilon_1{}^2+\epsilon_2{}^2)}{\epsilon_1\epsilon_2}\left(1-\frac{1}{\lambda}\right)\end{aligned}\right]$$

$$=\frac{k^2q^2}{\lambda^2}\left[\frac{(\epsilon_1\epsilon_2\lambda-k^2)}{\lambda^2}\left(-\frac{1}{\epsilon_1{}^2}-\frac{1}{\epsilon_2{}^2}+\frac{2}{\epsilon_1\epsilon_2}\right)-\frac{1}{\epsilon_1\epsilon_2\lambda}(\epsilon_1{}^2+\epsilon_2{}^2-2\epsilon_1\epsilon_2)+\frac{\epsilon_1{}^2+\epsilon_2{}^2}{\epsilon_1\epsilon_2}\right]$$

$$=\frac{k^2q^2}{\lambda^2}\left[-\frac{(\epsilon_1\epsilon_2\lambda-k^2)k^2}{\epsilon_1{}^2\epsilon_2{}^2\lambda^2}-\frac{k^2}{\epsilon_1\epsilon_2\lambda}+\frac{(\epsilon_1{}^2+\epsilon_2{}^2)}{\epsilon_1\epsilon_2}\right]$$

$$=\frac{k^2q^2}{\lambda^2}\left[-\frac{2k^2}{\epsilon_1\epsilon_2\lambda}+\frac{k^4}{\epsilon_1{}^2\epsilon_2{}^2\lambda^2}+\frac{(\epsilon_1{}^2+\epsilon_2{}^2)}{\epsilon_1\epsilon_2}\right]$$

$$=\frac{k^2q^2}{\lambda^2}\left[\frac{(\epsilon_1{}^2+\epsilon_2{}^2)}{\epsilon_1\epsilon_2}-\frac{k^2}{\lambda^2\epsilon_1\epsilon_2}\left(2\lambda-\frac{k^2}{\epsilon_1\epsilon_2}\right)\right].$$

Now for high energies (see (A4))

$$2\lambda \approx \frac{k^2}{\epsilon_1\epsilon_2} + 4\epsilon_1\epsilon_2\sin^2\tfrac{1}{2}\vartheta$$

and thus we have above, in (46),

$$\frac{k^2q^2}{\lambda^2}\left[\frac{(\epsilon_1{}^2+\epsilon_2{}^2)}{\epsilon_1\epsilon_2} - \frac{4k^2}{\lambda^2}\sin^2\tfrac{1}{2}\vartheta\right] . \tag{47}$$

We note that the terms in the factor [ ] here are identical to the terms on the right hand side of (45), apart from a sign. Thus, adding (45) and (47), we have, for the terms in { } in (2), the remarkably simple expression

$$\left\{\quad\right\} = \frac{k^2}{\lambda^2\epsilon_1\epsilon_2}\left[\epsilon_1{}^2+\epsilon_2{}^2 - \frac{4k^2\epsilon_1\epsilon_2}{\lambda^2}\sin^2\tfrac{1}{2}\vartheta\right]\left(q^2 - \frac{\lambda^2}{k^2}\right) . \tag{48}$$

We now write this in a form which is similar to that obtained for the screening correction considered in [6], where we integrated over final electron directions. For high energies we have, from (A4),

$$4\epsilon_1\epsilon_2\sin^2\tfrac{1}{2}\vartheta \approx 2\lambda - \frac{k^2}{\epsilon_1\epsilon_2}$$

and thus, in (48),

$$\frac{4k^2\epsilon_1\epsilon_2\sin^2\tfrac{1}{2}\vartheta}{\lambda^2} \approx \frac{k^2}{\lambda^2}\left(2\lambda - \frac{k^2}{\epsilon_1\epsilon_2}\right)$$

$$= \frac{2k^2}{\lambda}\left(1 - \frac{k^2}{2\epsilon_1\epsilon_2\lambda}\right)$$

$$\approx 4\epsilon_1\epsilon_2\frac{\delta k}{\lambda}\left(1 - \frac{\delta k}{\lambda}\right)$$

on substituting $\delta \approx \frac{k}{2\epsilon_1\epsilon_2}$ from (A1). The expression (48) for $\{\ \}$ can thus be written in the form

$$\{\ \} = \frac{k^2}{\lambda^2\epsilon_1\epsilon_2}\left[\epsilon_1{}^2+\epsilon_2{}^2 - 4\epsilon_1\epsilon_2\frac{\delta k}{\lambda}\left(1 - \frac{\delta k}{\lambda}\right)\right]\left(q^2 - \frac{\lambda^2}{k^2}\right) \quad . \tag{49}$$

Now from (A7) and (A19) we have, for high energies and small scattering angles,

$$\frac{\lambda}{k} \approx \delta(1 + w^2) \equiv \frac{\delta}{\chi} \tag{50}$$

where w is defined by (A6):

$$w = \frac{\epsilon_1\epsilon_2\vartheta}{k} \quad . \tag{51}$$

Substituting (50) in (49) then gives

$$\{\ \} = \frac{\chi^2}{\delta^2\epsilon_1\epsilon_2}\left[\epsilon_1{}^2+\epsilon_2{}^2 - 4\epsilon_1\epsilon_2\chi(1 - \chi)\right]\left(q^2 - \frac{\delta^2}{\chi^2}\right) \quad . \tag{52}$$

Finally, from the definition for χ, (A19),

$$\chi = \frac{1}{1 + w^2}$$

we have

$$\chi(1 - \chi) = w^2\chi^2 \quad ,$$

which, when substituted in (52), gives

$$\left\{\;\right\} = \frac{\chi^2}{\delta^2\epsilon_1\epsilon_2}\left[\epsilon_1{}^2+\epsilon_2{}^2 - 4\epsilon_1\epsilon_2 w^2\chi^2\right]\left(q^2 - \frac{\delta^2}{\chi^2}\right) . \tag{53}$$

In order to obtain the screening correction, we must substitute (53) in (1), and replace $(1 - F)^2$ in the integrand by $(1 - F)^2 - 1$, as indicated in (4) and (5). Further, in the lower limit in (1) we can write, from (A20),

$$q_m \approx \frac{\delta}{\chi} \tag{54}$$

and take the upper limit to be infinite since there is negligible contribution to the screeing correction from $q \gtrsim O(1)$. We then have

$$\frac{d\sigma_{scr\ corr}}{d\Omega} = \frac{1}{2\pi}\frac{e^2}{\hbar c}\left(\frac{Ze^2}{mc^2}\right)^2\frac{d\epsilon_2}{\epsilon_1{}^2k}\frac{\chi^2}{\delta^2}\left[\epsilon_1{}^2+\epsilon_2{}^2 - 4\epsilon_1\epsilon_2 w^2\chi^2\right]$$

$$\times \int_{\delta^2/\chi^2}^{\infty}\frac{[(1-F)^2 - 1]}{q^4}\left(q^2 - \frac{\delta^2}{\chi^2}\right)d(q^2) \tag{55}$$

In order to compare this expression with the screening correction given in [6] we note from (50) and (51) that for small scattering angles, ϑ,

$$
\begin{aligned}
d\Omega &= \sin\vartheta \; d\vartheta \; d\phi \\
&\approx \vartheta \; d\vartheta \; d\phi \\
&= \tfrac{1}{2}\left(\frac{k}{\epsilon_1 \epsilon_2}\right)^2 d(w^2) \; d\phi \\
&= 2\delta^2 d(w^2) \; d\phi \\
&= -2 \frac{\delta^2}{\chi^2} \, d\chi \; d\phi \quad . \qquad (56)
\end{aligned}
$$

Then, since the expression (55) is independent of ϕ, integration over this variable merely gives a factor 2π. We then have, if we take q rather than q^2 as the variable of integration,

$$
d\sigma_{\text{scr corr}} = 4 \frac{e^2}{\hbar c} \left(\frac{Ze^2}{mc^2}\right)^2 \frac{d\epsilon_2}{\epsilon_1{}^2 k} \, d\chi \left[\epsilon_1{}^2 + \epsilon_2{}^2 - 4\epsilon_1 \epsilon_2 w^2 \chi^2\right]
$$

$$
\times \int_{\delta/\chi}^{\infty} \frac{\left[(1-F)^2 - 1\right]}{q^3} \left(q^2 - \frac{\delta^2}{\chi^2}\right) dq \quad . \qquad (57)
$$

Comparison with the cross section integrated over final electron directions, i.e., the angular distribution of the photons, given in [6], p. 897, eq (7.2),

then shows that (57) is identical to the screening correction* in [6], once one has made the substitutions

$$dk \to d\epsilon_2$$

$$u \to w$$

$$\xi = \frac{1}{1 + u^2} \to \frac{1}{1 + w^2} = \chi \quad . \tag{58}$$

Having already noted that in the absence of recoil $dk = -d\epsilon_2$, the significant replacement in (58) is

$$u \to w$$

where

$$u = p_1\theta_1 \qquad \text{with} \qquad \theta_1 = \measuredangle\left(\underline{p}_1, \underline{k}\right)$$

is the convenient variable when the photon is observed, and

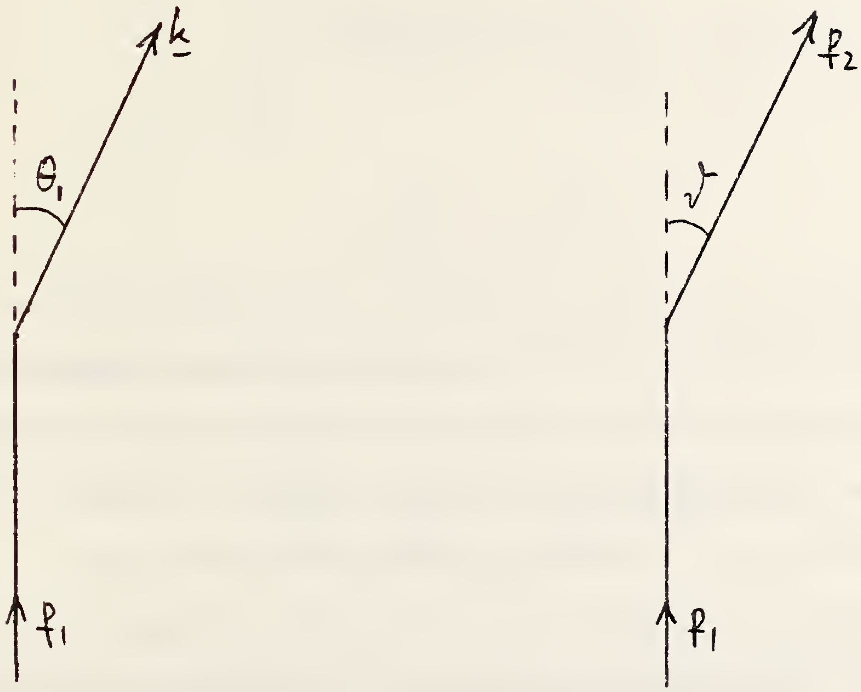
$$w = \frac{p_1 p_2 \vartheta}{k} \qquad \text{with} \qquad \vartheta = \measuredangle\left(\underline{p}_1, \underline{p}_2\right)$$

---

*Note that the screening correction in [6] is contained in the terms in (7.2) with factor Γ, defined in (6.29) and (6.28) in that reference. It comes, specifically, from the term $\mathcal{F}(\delta/\xi)$ in Γ, as can be seen from eq (6.28) there.

is the corresponding variable when the final electron is observed.

Indeed, a comparison of the two situations in the figure below



photon observed
integrate over electron angles

electron observed
integrate over photon angles

suggests that the analogous variables might be

$$k\theta_1 \qquad \text{(photon observed)}$$

and (59)

$$p_2\vartheta \qquad \text{(electron observed)} \quad .$$

This would imply the substitution given in (58), since from

$$k\theta_1 \to p_2\vartheta$$

we have

$$u = p_1\theta_1 = \frac{p_1}{k}(k\theta_1) \rightarrow \frac{p_1}{k}(p_2\vartheta) \approx \frac{\epsilon_1\epsilon_2\vartheta}{k} = w$$

and

$$\xi = \frac{1}{1+u^2} \rightarrow \frac{1}{1+w^2} = \chi \quad .$$

Having shown the identity of the screening correction derived here, eq (57), with that given earlier in [6] when integrating over electron angles, under the substitution (58), we choose finally to express it in terms of the variable q_m. It then has the same form whether one integrates over photon or electron angles, the only difference being the definition of the minimum momentum transfer, q_m, in terms of the remaining variables, viz.,

$$q_m = |\underline{p}_1 - \underline{k}| - p_2 \qquad \text{photon observed} \qquad (60a)$$

$$q_m = |\underline{p}_1 - \underline{p}_2| - k \qquad \text{final electron observed} \qquad (60b)$$

Substituting q_m as given in (54) into the expression (55) we have

$$\frac{d\sigma_{\text{scr corr}}}{d\Omega} = \frac{1}{2\pi}\,\frac{e^2}{\hbar c}\left(\frac{Ze^2}{mc^2}\right)^2 \frac{d\epsilon_2}{\epsilon_1{}^2 k}\left[\epsilon_1{}^2+\epsilon_2{}^2 - 4\epsilon_1\epsilon_2\,\frac{\delta}{q_m}\left(1 - \frac{\delta}{q_m}\right)\right]$$

$$\times \int_{q_m{}^2}^{\infty} \frac{\left[(1-F(q))^2 - 1\right]}{q^4}\left(\frac{q^2}{q_m{}^2} - 1\right) d(q^2) \quad . \tag{61}$$

From (4) and (61) we now have our complete expression for the cross section integrated over photon angles:

$$\frac{d\sigma_{\text{scr}}}{d\Omega} = \frac{d\sigma_{\text{unscr}}}{d\Omega} + \frac{d\sigma_{\text{scr corr}}}{d\Omega} \quad . \tag{62}$$

As noted earlier, the unscreened cross section (for bremsstrahlung in the field of a point Coulomb potential) can be integrated in closed form over the photon angles, and is given in [1], p. B1346, eq (6). The expression given in that reference is for arbitrary energies and angles, and is consequently rather lengthy. Since for our present considerations it is quite appropriate to assume $\epsilon_1 >> 1$ and $\epsilon_2 >> 1$, we give here the high energy form of that expression without, however, making the small angle approximation, $\vartheta << 1$. We then have

$$\frac{d\sigma_{unscr}}{d\Omega} = \frac{1}{2\pi}\,\frac{e^2}{\hbar c}\left(\frac{Ze^2}{mc^2}\right)^2 \frac{d\epsilon_2}{k\epsilon_1^2} \cdot 2\epsilon_1\epsilon_2 \cdot \left(\quad\right) \tag{63a}$$

where

$$\left(\quad\right) = \frac{k^2\lambda + 2\epsilon_1\epsilon_2(\lambda+1)\cos^2\tfrac{1}{2}\vartheta}{\lambda^2[\lambda(\lambda+2)]^{\frac{1}{2}}}\,\ell n\left\{\lambda + 1 + [\lambda(\lambda+2)]^{\frac{1}{2}}\right\}$$

$$+ \frac{k}{\lambda^2}\left[-k + \frac{\epsilon_1(\epsilon_1+\epsilon_2)}{\epsilon_2}\cos^2\tfrac{1}{2}\vartheta + \frac{\epsilon_1^2(2\epsilon_2-k)\sin^2\vartheta}{\lambda^2}\right]\ell n(2\epsilon_2)$$

$$- \frac{k}{\lambda^2}\left[k + \frac{\epsilon_2(\epsilon_1+\epsilon_2)}{\epsilon_1}\cos^2\tfrac{1}{2}\vartheta + \frac{\epsilon_2^2(2\epsilon_1+k)\sin^2\vartheta}{\lambda^2}\right]\ell n(2\epsilon_1)$$

$$+ \frac{k^2}{2\epsilon_1\epsilon_2\lambda}\left[\epsilon_1^2+\epsilon_2^2 - 2(\epsilon_1^2+\epsilon_2^2+\epsilon_1\epsilon_2)\cos^2\tfrac{1}{2}\vartheta\right]$$

$$+ \frac{2k^2\sin^2\vartheta}{\lambda^4}\left(\epsilon_1^2+\epsilon_2^2-\epsilon_1\epsilon_2\right) - \frac{2\epsilon_1\epsilon_2}{\lambda^2}\cos^2\tfrac{1}{2}\vartheta \quad . \tag{63b}$$

The high energy cross section for electrons which have scattered through an angle ϑ and radiated a photon, integrated over photon angles, is then given by eq (62), in which we substitute eqs (61) and (63). In these equations the variables δ, q_m, and λ are given by (A1), (A12), and (A4), viz.,

$$\delta = \frac{k}{2\epsilon_1\epsilon_2}$$

$$q_m = \frac{\lambda}{k}$$

$$\lambda = \frac{k}{2\epsilon_1\epsilon_2} + 2\epsilon_1\epsilon_2\sin^2\tfrac{1}{2}\vartheta \quad . \tag{64}$$

It is to be noted that although the screening correction, eq (61), has the same relatively simple form whether we integrate over photon or electron angles, the unscreened cross section is very different in the two cases. The rather simple form that results for high energies and small angles when integrating over electron angles ([5], p. 924, Formula 2BN(a)) is unfortunately not obtained when integrating over photon angles, as is clear from eq (63), even when we make the small angle approximation there, viz., by setting $\sin^2\vartheta \approx \vartheta^2$, $\cos^2\frac{1}{2}\vartheta \approx 1$. The rather complicated argument of the logarithm in (63) does not simplify for small angles, where $\lambda = O(1)$.

V. SCREENING CORRECTION IN THE THOMAS-FERMI-MOLIÈRE MODEL

Although the screening correction is given formally by eq (61) as an integral over the atomic form factor, it is clear that a simple closed form expression without integrals is useful for practical applications, particularly if one must later perform an integration over this cross section. We therefore perform the integration in (61) in closed form for the Thomas-Fermi model as used by Molière [7], viz., with

$$\frac{1-F(q)}{q^2} = \sum_{i=1}^{3} \frac{\alpha_i}{\beta_i^{\,2} + q^2} \tag{65}$$

where

$$\alpha_1 = 0.10 \quad , \quad \alpha_2 = 0.55 \quad , \quad \alpha_3 = 0.35$$

$$\beta_i = (Z^{1/3}/121)b_i; \quad b_1 = 6.0 \; , \; b_2 = 1.20 \; , \; b_3 = 0.30 \; . \tag{65a}$$

The result of this integration is then rather simple [6]:

Defining the function $\mathcal{C}(q_m)$ by

$$\int_{q_m^2}^{\infty} \frac{[(1-F(q))^2 - 1]}{q^4} \left(\frac{q^2}{q_m^2} - 1\right) d(q^2) \equiv - \frac{1}{q_m^2} \mathcal{C}(q_m) \tag{66}$$

we find

$$\mathcal{C}(q_m) = \sum_{i=1}^{3} \alpha_i^2 \, \ln(1+B_i)$$

$$-2\sum_{\substack{i=1 \\ i \neq j}}^{3} \sum_{j=1}^{3} \alpha_i \alpha_j \left[\frac{1+B_j}{B_i - B_j} \ln(1+B_j) + \frac{1}{2}\right] \tag{67}$$

where

$$B_i = \left(\frac{\beta_i}{q_m}\right)^2 \quad . \tag{68}$$

In the absence of screening, i.e., for $\frac{\beta_i}{q_m} \ll 1$, eq (67) gives

$$\mathcal{C}(q_m) = O\left(\left(\frac{\beta_i}{q_m}\right)^2\right) \ll 1 \quad , \tag{69}$$

as expected. For $\frac{\beta_i}{q_m} \gg 1$. i.e., for complete screening, eq (67) gives

$$C(q_m) = \ln\left[\left(\frac{Z^{1/3}}{111.8\ q_m}\right)^2\right] + 0\left(\frac{1}{\left(\frac{\beta_i}{q_m}\right)^2}\right) . \tag{70}$$

Thus, if we write, in the limit of complete screening,

$$C_{\text{compl.scr.}}(q_m) = \ln\left[1 + \left(\frac{Z^{1/3}}{111.8\ q_m}\right)^2\right] \tag{71}$$

then both the complete screening and the no screening limits, eqs (70) and (69), are given by the single expression, (71). A comparison of $C = C(q_m)$ as given by (67) with $C_{\text{compl.scr.}}$ as given by (71) shows that the relative error incurred by using (71), viz., $(C_{\text{compl. scr.}} - C)/C$, is less than 1% for $\frac{Z^{1/3}}{111.8\ q_m} > 6.4$ and less than 2% for $\frac{Z^{1/3}}{111.8\ q_m} > 4.7$.

For most practical applications requiring the screening correction, the expression given in (67) may be used for the integral in (61). Further, in the case of complete screeing, i.e., when $\frac{Z^{1/3}}{111.8\ q_m} \gtrsim 5$, one may use the even simpler expression given in (71).

APPENDIX

In this appendix we derive the high energy approximations used in the body of the report, in particular for the variables δ, λ_0, and λ defined in (3), and for the quantity $\left(\frac{\epsilon_1}{p_1} - \frac{\epsilon_2}{p_2}\right)^2$, which enters in eqs (31) and (32). In addition we derive approximations for λ and q_m, valid for both high energies and small scattering angle, ϑ.

By high energy approximation we mean $\epsilon_1 \gg 1$ and $\epsilon_2 \gg 1$, and imply neglect of terms of relative order $1/\epsilon_1^2$ and $1/\epsilon_2^2$, which we refer to as neglect of terms of relative order $1/\epsilon^2$. By small angle approximation we mean $\vartheta \ll 1$ and imply neglect of terms of relative order ϑ^2.

We start by deriving the high energy approximation for δ, given in (3):

$$\delta \equiv p_1 - p_2 - k$$

$$= p_1 - p_2 - \epsilon_1 + \epsilon_2$$

$$= (\epsilon_2 - p_2) - (\epsilon_1 - p_1)$$

$$= \frac{1}{\epsilon_2 + p_2} - \frac{1}{\epsilon_1 + p_1} \qquad \text{(since } \epsilon^2 - p^2 = 1\text{)}$$

$$= \frac{\epsilon_1 + p_1 - \epsilon_2 - p_2}{(\epsilon_1 + p_1)(\epsilon_2 + p_2)} \quad .$$

Here

$$p_1-p_2 = \frac{(p_1^2-p_2^2)}{p_1+p_2}$$

$$= \frac{(\epsilon_1^2-\epsilon_2^2)}{p_1+p_2}$$

$$= (\epsilon_1-\epsilon_2)\,\frac{(\epsilon_1+\epsilon_2)}{(p_1+p_2)}$$

$$= k\,\frac{(\epsilon_1+\epsilon_2)}{(p_1+p_2)} \quad .$$

Thus,

$$\delta = \frac{k\left(1 + \dfrac{\epsilon_1+\epsilon_2}{p_1+p_2}\right)}{(\epsilon_1+p_1)(\epsilon_2+p_2)}$$

$$= \frac{k(\epsilon_1+p_1+\epsilon_2+p_2)}{(p_1+p_2)(\epsilon_1+p_1)(\epsilon_2+p_2)} \quad .$$

Thus far we have made no approximations, and have written δ in a form in which there are no cancellations. We can now set $p_1 = \epsilon_1(1+0(1/\epsilon_1^2))$ and $p_2 = \epsilon_2(1+0(1/\epsilon_2^2))$ and find, neglecting terms of relative order $1/\epsilon_1^2$ and $1/\epsilon_2^2$,

$$\delta \approx \frac{k}{2\epsilon_1\epsilon_2} \quad . \qquad \text{(A1)}$$

Next we consider (see (3))

$$\begin{aligned}
\lambda_0 &= \epsilon_1\epsilon_2 - p_1p_2 - 1 \\
&= \tfrac{1}{2}[(p_1-p_2)^2 - (\epsilon_1-\epsilon_2)^2] \\
&= \tfrac{1}{2}[(p_1-p_2)^2 - k^2] \\
&= \tfrac{1}{2}(p_1-p_2-k)\ (p_1-p_2+k) \\
&= \tfrac{1}{2}\delta(p_1-p_2-k+2k) \\
&= \delta(k + \tfrac{1}{2}\delta) \quad .
\end{aligned}$$

Again we have an expression with no cancellations. We can now use our high energy approximation for δ, eq (A1), and obtain

$$\begin{aligned}
\lambda_0 &\approx \frac{k^2}{2\epsilon_1\epsilon_2} + \frac{k^2}{8\epsilon_1{}^2\epsilon_2{}^2} \\
&= \frac{k^2}{2\epsilon_1\epsilon_2}\left(1 + \frac{1}{4\epsilon_1\epsilon_2}\right) .
\end{aligned}$$

The second term is of relative order $1/\epsilon^2$ and is thus neglected, giving the high energy approximation

$$\lambda_0 \approx \frac{k^2}{2\epsilon_1\epsilon_2} \quad . \tag{A2}$$

Next, in eqs (31) and (32) we have the factor

$$\left(\frac{\epsilon_1}{p_1} - \frac{\epsilon_2}{p_2}\right)^2 = \frac{(\epsilon_1 p_2 - p_1 \epsilon_2)^2}{p_1^2 p_2^2}$$

Here

$$(\epsilon_1 p_2 - p_1 \epsilon_2)^2 = \epsilon_1^2 \epsilon_2^2 - \epsilon_1^2 - 2p_1 p_2 \epsilon_1 \epsilon_2 + p_1^2 p_2^2 + p_1^2$$

$$= (\epsilon_1 \epsilon_2 - p_1 p_2)^2 - 1$$

$$= (\epsilon_1 \epsilon_2 - p_1 p_2 - 1)(\epsilon_1 \epsilon_2 - p_1 p_2 + 1)$$

$$= \lambda_0(\lambda_0 + 2) \quad .$$

Using the high energy approximation for λ_0 just given in (A2) we have

$$(\epsilon_1 p_2 - p_1 \epsilon_2)^2 \approx \frac{k^2(k^2 + 4\epsilon_1 \epsilon_2)}{(2\epsilon_1 \epsilon_2)^2}$$

$$= \frac{k^2(\epsilon_1 + \epsilon_2)^2}{(2\epsilon_1 \epsilon_2)^2} \quad .$$

We thus have the high energy approximation

$$\left(\frac{\epsilon_1}{p_1} - \frac{\epsilon_2}{p_2}\right)^2 \approx \frac{k^2(\epsilon_1 + \epsilon_2)^2}{4\epsilon_1^4 \epsilon_2^4} \tag{A3}$$

which is given in eqs (31) and (32) as

$$\left(\frac{\epsilon_1}{p_1} - \frac{\epsilon_2}{p_2}\right)^2 = 0\left(\frac{k^2}{\epsilon^6}\right) . \tag{A3a}$$

Next we have the important variables λ and q_m. From (3)

$$\lambda = \epsilon_1\epsilon_2 - p_1p_2\cos\vartheta - 1$$

$$= \epsilon_1\epsilon_2 - p_1p_2 - 1 + p_1p_2(1-\cos\vartheta)$$

$$= \lambda_0 + 2p_1p_2\sin^2\tfrac{1}{2}\vartheta \quad .$$

Thus from (A2) we have, at high energies (but arbitrary angle ϑ),

$$\lambda \approx \frac{k^2}{2\epsilon_1\epsilon_2} + 2\epsilon_1\epsilon_2\sin^2\tfrac{1}{2}\vartheta \quad . \tag{A4}$$

If in addition $\vartheta \ll 1$, as it is in the screening correction, then we have

$$\lambda \approx \frac{k^2}{2\epsilon_1\epsilon_2} + \frac{\epsilon_1\epsilon_2\vartheta^2}{2}$$

$$= \frac{k^2}{2\epsilon_1\epsilon_2}\left[1 + \left(\frac{\epsilon_1\epsilon_2\vartheta}{k}\right)^2\right] .$$

From (A1) this can be written as

$$\lambda \approx k\delta\left[1 + \left(\frac{\epsilon_1\epsilon_2\vartheta}{k}\right)^2\right] . \tag{A5}$$

This suggests that we define the variable w by

$$w = \frac{\epsilon_1 \epsilon_2 \vartheta}{k} \tag{A6}$$

from which

$$\lambda \approx k\delta(1 + w^2)$$

or

$$\frac{\lambda}{k} \approx \delta(1 + w^2) \quad . \tag{A7}$$

Equation (A7) provides a useful expression for the minimum momentum transfer, q_m, in the region of importance for the screening correction, $q_m \lesssim O(\beta)$, since this condition, together with the assumption of high energies, implies small scattering angles. This may be seen from q_m as defined in (3): Starting with

$$q_m = (2\lambda + k^2)^{\frac{1}{2}} - k \lesssim O(\beta)$$

we write this as

$$(2\lambda + k^2)^{\frac{1}{2}} - k \leqq c\beta$$

where c is a constant of order unity. Thus

$$(2\lambda + k^2)^{\frac{1}{2}} \leqq k + c\beta \quad .$$

Squaring both sides we then have

$$2\lambda \leqq 2ck\beta + c^2\beta^2$$

which may be written as

$$\lambda \lesssim O(k\beta) + O(\beta^2) \quad .$$

With λ as given at high energies in (A4), we have

$$\lambda \approx \frac{k^2}{2\epsilon_1\epsilon_2} + 2\epsilon_1\epsilon_2\sin^2\tfrac{1}{2}\vartheta \lesssim O(k\beta) + O(\beta^2)$$

and hence

$$2\epsilon_1\epsilon_2\sin^2\tfrac{1}{2}\vartheta \lesssim O(k\beta) + O(\beta^2)$$

or

$$\sin^2\tfrac{1}{2}\vartheta \lesssim O\left(\frac{k\beta}{\epsilon^2}\right) + O\left(\frac{\beta^2}{\epsilon^2}\right) \lesssim O\left(\frac{\beta}{\epsilon}\right) << 1 \quad . \tag{A8}$$

Thus the condition $q_m \lesssim O(\beta)$ (together with tha assumption $\epsilon_1 >> 1$, $\epsilon_2 >> 1$) implies small scattering angles

$$\vartheta^2 \lesssim O\left(\frac{\beta}{\epsilon}\right) \tag{A9}$$

for all photon energies, k. (In the body of this report we derived this result for $k = O(\epsilon)$; see eq (17).) We may thus use (A7) to obtain our expression for q_m in the region $q_m \lesssim O(\beta)$. Again referring to (3), we have

$$q_m = (2\lambda + k^2)^{\frac{1}{2}} - k$$

from which

$$q_m^{\,2} + 2kq_m = 2\lambda$$

or

$$q_m = \frac{\frac{\lambda}{k}}{1 + \frac{q_m}{2k}} \quad . \tag{A10}$$

Thus if we assume that the photons are not too soft, specifically, provided that*

$$2k \gg q_m \tag{A11}$$

then

$$q_m \approx \frac{\lambda}{k} \tag{A12}$$

---

*Note that since $q_m \lesssim O(\beta)$, the condition $2k \gg q_m$ will be satisfied provided $2k \gg \beta \sim \frac{Z^{1/3}}{111.8} \lesssim \frac{1}{25}$. Thus to have $2k \gg q_m$, it is sufficient to have $k \gg \frac{1}{50}\, mc^2 \approx 10$ keV.

and from (A7)

$$q_m \approx \delta(1 + w^2) \quad . \tag{A13}$$

It is worth noting the obvious similarity between (A13) and the high energy, small angle approximation for the minimum momentum transfer when one integrates over the final electron directions [6]. In that case

$$q_m = |\underline{p}_1 - \underline{k}| - p_2$$

$$\approx \delta(1 + u^2) \tag{A14}$$

where

$$u = p_1\theta_1 \tag{A15}$$

with

$$\theta_1 = \measuredangle\, (\underline{p}_1,\underline{k}) \quad . \tag{A16}$$

When integrating over final electron directions [6] it was convenient to define

$$\xi = \frac{1}{1 + u^2} \tag{A17}$$

in terms of which

$$q_m = \frac{\delta}{\xi} \ . \tag{A18}$$

In the present situation, in which we integrate over photon directions, we define

$$\chi = \frac{1}{1 + w^2} \tag{A19}$$

in terms of which

$$q_m = \frac{\delta}{\chi} \ . \tag{A20}$$

Although in [6] the screening correction was expressed in terms of the variables u and ξ, we note in the body of this report (see eq (61)) that the two screening corrections have the same form, identically, when expressed in terms of q_m, defined by $q_m = |\underline{p}_1 - \underline{k}| - p_2$ when the photon is observed, and by $q_m = |\underline{p}_1 - \underline{p}_2| - k$ when the final electron is observed.

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We derive an expression for the angular distribution of high energy electrons which have undergone scattering and radiated a photon, integrated over the directions of the emitted photon, in the region of small scattering angles, for which the atomic form factor must be taken into account, but the nuclear structure may be neglected. This distribution is analogous to Schiff's high energy small angle expression for the angular distribution of photons, integrated over final electron angles. We show that the correction to the angular distribution of electrons due to atomic screening is identical in form to the correction for the angular distribution of photons. This correction involves an integral over the atomic form factor, which is evaluated in closed form for the Thomas-Fermi-Moilere model. A very simple expression is obtained for the case of complete screening.

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